

A double copy for $\mathcal{N} = 2$ supergravity: a linearised tale told on-shell

G.L. Cardoso, S. Nagy and S. Nampuri

*Center for Mathematical Analysis, Geometry and Dynamical Systems,
Department of Mathematics, Instituto Superior Técnico,
Universidade de Lisboa, Av. Rovisco Pais, 1049-001 Lisboa, Portugal*

gcardoso@math.tecnico.ulisboa.pt , snagy@math.tecnico.ulisboa.pt , nampuri@gmail.com

ABSTRACT

We construct the on-shell double copy dictionary for linearised four-dimensional $\mathcal{N} = 2$ supergravity coupled to one vector multiplet with a quadratic prepotential. We apply this dictionary to the weak-field approximation of dyonic BPS black holes in this theory.

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1 Introduction

One of the richest and most outstanding pursuits in gravitation for the past decades has been the attempt to formulate gravity in terms of gauge field degrees of freedom. One approach towards this goal is the gauge-gravity duality approach which seeks to describe gravitational degrees of freedom as holographically encoded in terms of a lower dimensional field theory. The AdS/CFT correspondence [1–3], which equates the gravitational path integral in the bulk with given boundary conditions for the fields to the path integral in a lower dimensional CFT, is the pinnacle of achievement based on this theme. A more kinematically flavored approach is based on rewriting gravity amplitudes as double copies of gauge theory amplitudes [4–12]. The double copy approach¹ indicates that heuristically, to rewrite gravity in terms of squared gauge theories, one must replace gravitational fields by a tensor product of appropriate gauge fields terms [14–20]. The holographic approach has yielded invaluable insights into the non-perturbative structure of gravity in terms of

¹See [13] for a review.

the organization of dual CFT data, but it applies to spacetimes that are asymptotically AdS. The second approach yields the rewriting of gravitational amplitudes in flat spacetime, in terms of gauge theory amplitudes. However, it has not yet produced direct knowledge of solitonic configurations, as computing amplitudes in non-trivial backgrounds remains a mathematically challenging exercise, to date. Motivated by these considerations, here, we adopt the 'Gravity as a double copy of gauge theories' philosophy inspired by the second approach and initiate a program to develop a prescription for mapping on-shell configurations of $D = 4, \mathcal{N} = 2$ ungauged supergravity theories to a double copy description. For the purposes of this note, which is to demonstrate the existence of such a consistent lexicon, we restrict ourselves to one of the simplest $\mathcal{N} = 2$ supergravity theories, describing the coupling of one vector multiplet to supergravity, namely the one based on a quadratic prepotential given by $F = -iX^0X^1$.

The first step here is to verify a match between the on-shell degrees of freedom in gauge and gravity theories. The next step is to propose a mathematical structure that allows gravitational fields to be written as tensor-like combinations of the gauge fields, and which naturally incorporate a map between symmetries on the gravity side, such as diffeomorphism, to local gauge symmetries in the double copy description [18]. At the linearised approximation level of gravity, inspired by the results of the amplitude calculations which indicate that fields like the graviton should be replaced by a tensor product of gauge fields in momentum space, [18] proposed an ansatz in position space that maps a linearised fluctuation in the gravity theory to a convolution of two fields, one from each gauge theory of the double copy. Thus the linearised fluctuation of a gravitational field configuration, Φ_G , will have a double copy description,

$$\Phi_G(x) = [\phi \star \tilde{\phi}](x) = \int \phi(y) \tilde{\phi}(x-y) dy, \quad (1.1)$$

where the \star denotes a convolution, and where ϕ and $\tilde{\phi}$ denote field theory configurations. Linearised gauge transformations of the field quantities on the right hand side in the above equation result in linearised local symmetry transformations on the gravity side. Further, differential operators acting on the double copy convolution (1.1) are allowed to hit either of the terms in the convolution. For the model discussed in this paper, the field theory containing ϕ exhibits the same number of supersymmetries (namely $\mathcal{N} = 2$) as the gravity theory, while the field theory containing $\tilde{\phi}$ carries no supercharges. Therefore, suppose that one establishes an ansatz for any given on-shell Φ_G such that local symmetries on both sides of the double copy equality are mapped to each other. Then, the on-shell differential operator D which annihilates Φ_G to form its equation of motion $D\Phi_G = 0$, annihilates the convolution (1.1) of the two corresponding fields in the double copy via the field theory equations of motion. Under a linearised supersymmetry transformation acting on the gravitational field, the supersymmetry transformation also acts on the corresponding field in the supersymmetric theory on the double copy side. On the gravity side, under this supersymmetry transformation one gets a new on-shell field. The same is true for the supersymmetric field theory in the double copy. Hence, the newly produced on-shell field configurations in the gravity and gauge sectors must be mapped to each other by a consistent

dictionary. This must be true for all states that are generated down the supersymmetry ladder, and hence one expects the dictionary to be consistently established for all on-shell fields. In this paper we check this argument for linearised $\mathcal{N} = 2$ supergravity coupled to one vector multiplet with prepotential $F = -iX^0X^1$.

The idea of developing a double copy dictionary for non-supersymmetric gravitational solutions has been pursued recently in [21–24]. In this note we will address a similar question for gravitational BPS solitons.

We start with a double copy ansatz for a combination of the gravitini and gaugini. By repeated application of the linearised supersymmetry transformations, we derive the on-shell dictionary. This dictionary is valid for all long multiplet configurations. However, for short (BPS) configurations, which have vanishing fermionic fields, the full dictionary cannot be generated by supersymmetry transformations in the procedure described above. Hence, the proof of the double copy prescription is a priori not valid for such states. Also, these BPS configurations are sourced, while the dictionary we construct is for on-shell source free configurations. However, this mirrors the case of on-shell linearised supergravity theories which are constructed in source free set-ups, but whose equations of motion generate sourced solitonic states. So, the validity of this dictionary for BPS states cannot be trivially discounted. Therefore, we test the dictionary empirically on these states by applying it to the weak-field approximation of dyonic (carrying both electric and magnetic charges) BPS black holes in this theory, and find that it holds. Hence we conjecture that the dictionary holds for all linearised BPS on-shell configurations in this model. We conclude with some comments on technical caveats in the dictionary and point out the next steps in the program.

2 Double copy dictionary

We construct the on-shell double copy dictionary for one of the simplest four-dimensional $\mathcal{N} = 2$ supergravity theories with vector multiplets, namely supergravity coupled to one vector multiplet based on the prepotential $F = -iX^0X^1$. We use the superconformal approach to $\mathcal{N} = 2$ supergravity [25–29]. A brief summary of some of its features can be found in [Appendix A](#).

Traditionally, the on-shell double copy dictionaries given in the literature are in terms of momentum states (see [4, 10, 11, 17, 19, 30–32] for some examples). Here, we will derive a double copy dictionary in position space by means of the convolution (1.1).

For the model at hand, the double copy construction proceeds by tensoring an $\mathcal{N} = 2$ super Yang-Mills multiplet with an ($\mathcal{N} = 0$) gauge field. At the level of momentum states, it was shown in [19] that these are the multiplets that are relevant for the double copy construction of this model. This is displayed in [Table 1](#), where we give the helicity eigenstates that result from the tensoring. Since the Yang-Mills fields may lie in a representation of a global non-Abelian group, an additional spectator field will have to be included in the dictionary, leading to a generalization of (1.1) [18].

The double copy dictionary is a dictionary for fluctuations around a fixed background. On the supergravity side, we take the background to be given by flat spacetime, allowing for the presence of constant scalar fields which we denote by $\langle X^I \rangle$ ($I = 0, 1$). On the

	\tilde{A}^-	\tilde{A}^+
A^-	g^-	φ_0
λ_i^-	ψ_i^-	χ_i^+
σ^+, σ^-	$A_{0,1}^-$	$A_{0,1}^+$
λ_i^+	χ_i^-	ψ_i^+
A^+	φ_1	g^+

Table 1: On-shell $(\mathcal{N} = 2)_{SYM} \times (\mathcal{N} = 0)_{SYM} = (\mathcal{N} = 2)_{sugra} + (\mathcal{N} = 2)_{SYM}$

super Yang-Mills side, the background is also taken to be flat spacetime. We then derive the double copy dictionary by linearising these supersymmetric theories around these backgrounds. To keep the local symmetries manifest, we work with the corresponding gauge invariant quantities, i.e. field strengths, on the gravity side, to exhibit the double copy dictionary for these quantities.

In the following, we begin by reviewing the convolution structure in the presence of the aforementioned spectator field, and we discuss restrictions on the convolution integrals imposed by the equations of motion. Next, we display the linearised supersymmetry transformation rules that we will use to generate the double copy dictionary for all the fields involved. Then, we proceed to explain our double copy ansatz. Finally, we use the linearised supersymmetry transformation laws to work out the double copy relations for the supergravity fields. We verify that the linearised supersymmetry transformations on the super Yang-Mills side reproduce the linearised supergravity transformation rules. We refer to [Appendix D](#) for a detailed derivation of the double copy dictionary. Our on-shell dictionary is summarized in (2.28).

2.1 Convolution structure

Following [18], we allow the two fields that appear in the convolution integral (1.1) to transform in the adjoint representation of non-Abelian global groups G and \tilde{G} , respectively. Since the supergravity fields we will obtain through the double copy construction do not transform under these global transformations, a bi-adjoint spectator field $\phi_{a\tilde{a}}$ will have to be introduced into (1.1) so as to obtain a combination that is inert under global G (\tilde{G}) transformations. Thus, rather than working with (1.1), we will base our dictionary on the convolution structure [18]

$$\varphi_{sugra} = \varphi_{SYM}^a \star \phi_{a\tilde{a}} \star \tilde{\varphi}_{YM}^{\tilde{a}}, \quad (2.1)$$

where the indices a, \tilde{a} denote adjoint indices. The real scalar $\phi_{a\tilde{a}}$ transforms in the bi-adjoint of $G \times \tilde{G}$,

$$\delta\phi_{a\tilde{a}} = -f_{ac}^b \phi_{b\tilde{a}} \theta^c - f_{\tilde{a}\tilde{c}}^{\tilde{b}} \phi_{a\tilde{b}} \theta^{\tilde{c}}. \quad (2.2)$$

This scalar also appeared in the context of double copies in scattering amplitudes in [33, 34] and in supergravity solutions in [21–23]. In addition to these global transformations, the (super) Yang-Mills gauge fields in (2.1) also transform under local Abelian gauge transformations with parameters $\alpha^a(x)$ and $\tilde{\alpha}^{\tilde{a}}(x)$, respectively.

In (2.1), \star denotes the convolution

$$[f \star g](x) = \int d^4y f(y)g(x-y) . \quad (2.3)$$

This is an associative operation, which doesn't satisfy the Leibniz rule, but instead satisfies

$$\partial_\mu(f \star g) = (\partial_\mu f) \star g = f \star (\partial_\mu g) . \quad (2.4)$$

We will make extensive use of this property² when imposing equations of motion on both sides of the double copy relation (2.1), as well as when checking the transformation behaviour of both sides under linearised supersymmetry. Specifically, we will find that when imposing equations of motion on (2.1), we are led to constraints of the form

$$\partial^\mu \left(\varphi_{SYM}^a \star \phi_{a\tilde{a}} \star \tilde{A}_\mu^{\tilde{a}} \right) = 0 , \quad (2.5)$$

where φ_{SYM}^a is composed of fields from the $\mathcal{N} = 2$ super Yang-Mills multiplet. Using (2.4), this equals

$$\varphi_{SYM}^a \star \phi_{a\tilde{a}} \star \partial^\mu \tilde{A}_\mu^{\tilde{a}} = 0 , \quad (2.6)$$

which is automatically satisfied if we work in the Lorentz like gauge

$$\partial^\mu \tilde{A}_\mu^{\tilde{a}} = 0 . \quad (2.7)$$

Under a local Abelian transformation $\tilde{A} \rightarrow \tilde{A} + d\tilde{\alpha}$, we find the restriction (from (2.5))

$$\varphi_{SYM}^a \star \phi_{a\tilde{a}} \star \square \tilde{\alpha}^{\tilde{a}} = 0 , \quad (2.8)$$

which becomes $\square \tilde{\alpha}^{\tilde{a}} = 0$ in the gauge (2.7).

For simplicity, and without loss of generality, we will develop the double copy dictionary in the gauge (2.7).

2.2 Linearised transformation laws

In a double copy relation such as (2.1), subjecting fields on one of the sides to a transformation, induces a transformation of the fields on the other side. Thus, subjecting the fields on the right hand side to supersymmetry transformations will induce a supersymmetry transformation of the supergravity fields on the left hand side, and vice versa. This will be exploited below to construct the double copy dictionary for the supergravity theory based on the prepotential $F = -iX^0X^1$.

²Note that (2.4) holds in Cartesian coordinates, and hence we will present the double copy dictionary in these coordinates.

Thus, let us display the linearised transformation laws which we will be using in the following when setting up the double copy dictionary. Consider first a $\mathcal{N} = 2$ super Yang-Mills multiplet lying in a representation of a global non-Abelian group G . It transforms as follows under rigid supersymmetry (ϵ), local Abelian ($\alpha(x)$) and global non-Abelian (θ) transformations,

$$\begin{aligned}\delta A_\mu^a &= \left(\frac{1}{2} \varepsilon^{ij} \bar{\epsilon}_i \gamma_\mu \lambda_j^a + h.c. \right) + \partial_\mu \alpha^a + f_{bc}^a A_\mu^b \theta^c, \\ \delta \lambda_i^a &= \gamma^\mu \partial_\mu \sigma^a \epsilon_i + \frac{1}{4} \gamma^{\mu\nu} F_{\mu\nu}^{a-} \varepsilon_{ij} \epsilon^j + f_{bc}^a \lambda_i^b \theta^c, \\ \delta \sigma^a &= \frac{1}{2} \bar{\epsilon}^i \lambda_i^a + f_{bc}^a \phi^b \theta^c,\end{aligned}\tag{2.9}$$

where f_{bc}^a denote the structure constants of the global non-Abelian group G .

Note that the bosonic transformations above can be seen as the linearisation of the transformations corresponding to a local non-Abelian gauge group [18]. In this sense, we are mapping linearised super Yang-Mills theory to linearised supergravity.

An $\mathcal{N} = 0$ gauge field transforms as

$$\delta \tilde{A}_\mu^{\tilde{a}} = \partial_\mu \tilde{\alpha}^{\tilde{a}} + \tilde{f}_{\tilde{b}\tilde{c}}^{\tilde{a}} \tilde{A}_\mu^{\tilde{b}} \tilde{\theta}^{\tilde{c}},\tag{2.10}$$

where the global non-Abelian group \tilde{G} (with structure constants $\tilde{f}_{\tilde{b}\tilde{c}}^{\tilde{a}}$) may be different from the global group G above. Using these transformations laws together with (2.2), the convolution (2.1) is indeed inert under global $G \times \tilde{G}$ transformations.

Next, let us consider the linearised supersymmetry transformation rules for the fields appearing in the $\mathcal{N} = 2$ supergravity theory based on the prepotential $F = -iX^0 X^1$. As mentioned above, we linearise around a flat spacetime background with metric $\eta_{\mu\nu}$ and constant scalar fields $\langle X^I \rangle$, and hence, we linearise the spacetime metric and the scalar fields as

$$\begin{aligned}g_{\mu\nu} &= \eta_{\mu\nu} + h_{\mu\nu}, \\ X^I &= \langle X^I \rangle + \delta X^I.\end{aligned}\tag{2.11}$$

The physical scalar field $z = X^1/X^0$ is then linearised as

$$z = \frac{X^1}{X^0} = \langle z \rangle + \delta z, \quad \langle z \rangle = \frac{\langle X^1 \rangle}{\langle X^0 \rangle}, \quad \delta z = \frac{1}{\langle X^0 \rangle} (\delta X^1 - \langle z \rangle \delta X^0).\tag{2.12}$$

In the following, in order to avoid cluttering of notation, we will denote the fluctuations δX^I simply by X^I .

Linearising the supergravity transformation rules summarized in [Appendix B](#), we obtain for the model based on $F = -iX^0 X^1$ the following Q-supersymmetry transformation rules (dropping pure gauge terms in the variation of the gravitini),

$$\begin{aligned}
\delta_Q h_{\mu\nu} &= \bar{\epsilon}^i \gamma_{(\mu} \psi_{\nu)i} + h.c. , \\
\delta_Q \psi_\mu^i &= -\frac{1}{4} \gamma^{ab} \partial_{[a} h_{b]\mu}^- \epsilon^i - \frac{1}{16} T_{\alpha\beta}^- \gamma^{\alpha\beta} \gamma_\mu \epsilon^{ij} \epsilon_j , \\
\delta_Q W_\mu^0 &= \frac{1}{2} \epsilon^{ij} \bar{\epsilon}_i \gamma_\mu \Omega_j^0 + \epsilon^{ij} \bar{\epsilon}_i \psi_{\mu j} \langle X^0 \rangle + h.c. , \\
\delta_Q W_\mu^1 &= -\frac{\langle \bar{z} \rangle}{2} \epsilon^{ij} \bar{\epsilon}_i \gamma_\mu \Omega_j^0 + \langle z \rangle \epsilon^{ij} \bar{\epsilon}_i \psi_{\mu j} \langle X^0 \rangle + h.c. , \\
\delta_Q \Omega^{0i} &= \gamma^\mu \partial_\mu \bar{X}^0 \epsilon^i + \frac{1}{4} \gamma^{\mu\nu} \mathcal{F}_{\mu\nu}^{0+} \epsilon^{ij} \epsilon_j , \\
\delta_Q X^I &= \frac{1}{2} \bar{\epsilon}^i \Omega_i^I ,
\end{aligned} \tag{2.13}$$

where we used the gauge fixing condition for S-supersymmetry,

$$\Omega_i^1 = -\langle \bar{z} \rangle \Omega_i^0 , \tag{2.14}$$

to express Ω_i^1 in terms of Ω_i^0 . In the above, \pm denote the (anti)selfdual parts, and the composite quantities $T_{\mu\nu}^-$ and $\mathcal{F}_{\mu\nu}^{0+}$ are given by

$$\begin{aligned}
T_{\mu\nu}^- &= \frac{1}{\langle \bar{X}^0 \rangle} \left[F_{\mu\nu}^{0-} + \frac{F_{\mu\nu}^{1-}}{\langle \bar{z} \rangle} \right] , \\
\mathcal{F}_{\mu\nu}^{0+} &= \frac{1}{2} \left[F_{\mu\nu}^{0+} - \frac{F_{\mu\nu}^{1+}}{\langle z \rangle} \right] .
\end{aligned} \tag{2.15}$$

2.3 Dictionary

Now we derive the on-shell double copy dictionary for linearised supergravity based on $F = -iX^0 X^1$. The dictionary is summarized in (2.28).

To avoid ambiguities arising from gauge degrees of freedom when going on-shell, we will work with field strengths in our dictionary. Thus, rather than working with the metric fluctuation $h_{\mu\nu}$ we will work with the linearised Riemann tensor,

$$R_{\rho\sigma\mu\nu} = -2 \partial_{[\mu} \partial_{[\rho} h_{\sigma]\nu]} , \tag{2.16}$$

which is invariant under linearised diffeomorphisms,

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu . \tag{2.17}$$

Similarly, we work with $\psi_{\mu\nu}^i = 2\partial_{[\mu} \psi_{\nu]}^i$ for the gravitini, and so on.

Our strategy consists of postulating the following double copy ansatz for a linear combination of the supergravity fermions,

$$a\psi_{\mu\nu}^i + 2b\gamma_{[\nu} \partial_{\mu]} \Omega^{0i} \equiv \epsilon^{ij} \lambda_j^a \star \phi_{a\bar{a}} \star \tilde{F}_{\mu\nu}^{\bar{a}} , \tag{2.18}$$

where $a, b \in \mathbb{C}$ denote complex constants. Note that in view of (2.14), the left hand side of (2.18) captures all the relevant fermionic supergravity degrees of freedom. Inspection of Table 1 shows that this ansatz is the most general one compatible with Table 1.

At this point we recall that the fermionic fields appearing in (2.18) carry different weights under the $U(1)$ subgroup of the R-symmetry group (the so-called chiral weights). The left-handed gravitino ψ_μ^i has a chiral weight that differs by one unit from the chiral weight of the right-handed gaugino Ω^{0i} . This implies that we assign a zero chiral weight to a , while b will have to carry a compensating chiral weight so as to make $b\Omega^{0i}$ have the same chiral weight as $a\psi_\mu^i$ and λ_j^a .

Since, in the following, the spectator field $\phi_{a\bar{a}}$ will only play a passive role, we will omit its presence, to keep the expressions as simple as possible, and only reinstate its dependence at the end. Thus, we will for the time being suppress the non-Abelian indices and work with the double copy ansatz

$$a\psi_{\mu\nu}^i + 2b\gamma_{[\nu}\partial_{\mu]}\Omega^{0i} \equiv \varepsilon^{ij}\lambda_j \star \tilde{F}_{\mu\nu} . \quad (2.19)$$

By contracting (2.19) with γ^μ and using the equations of motion for the λ_i and ψ_μ^i (see Appendix D) as well as the property (2.4), we extract the dictionary for Ω^{0i} ,

$$2b\partial_\nu\Omega^{0i} = \varepsilon^{ij}\gamma^\mu\lambda_j \star \partial_\nu\tilde{A}_\mu , \quad (2.20)$$

and using this in (2.19) we infer the dictionary for the gravitini field strength,

$$2a\partial_{[\mu}\psi_{\nu]}^i = \varepsilon^{ij}\gamma^\rho\gamma_{[\nu}\lambda_j \star \partial_{\mu]}\tilde{A}_\rho . \quad (2.21)$$

These dictionary expressions have to be consistent with the linearised equations of motion for Ω^{0i} and for the gravitini. This is indeed the case, as we show in Appendix D.

Next, we return to (2.19) and verify its consistency with the equation of motion for \tilde{A}_μ , by acting with ∂^μ on (2.19). This results in

$$a(\Box\psi_\nu^i - \partial_\nu\partial^\mu\psi_\mu^i) = 0 , \quad (2.22)$$

where we used the equation of motion for Ω^{0i} . To verify that (2.22) vanishes, we take the equations of motion for the gravitini in the form $\gamma^\mu\partial_{[\mu}\psi_{\nu]}^i = 0$, and contract it with $\gamma^\rho\partial_\rho$,

$$\begin{aligned} 0 &= \gamma^\rho\gamma^\mu[\partial_\rho\partial_\mu\psi_\nu^i - \partial_\rho\partial_\nu\psi_\mu^i] \\ &= \gamma^{\rho\mu}[\partial_\rho\partial_\mu\psi_\nu^i - \partial_\rho\partial_\nu\psi_\mu^i] + \Box\psi_\nu^i - \partial^\mu\partial_\nu\psi_\mu^i \\ &= \Box\psi_\nu^i - \partial_\nu\partial^\mu\psi_\mu^i , \end{aligned} \quad (2.23)$$

where to get to the last line we used a consequence of the gravitini equation of motion, namely $\gamma^{\nu\rho}\partial_\nu\psi_\rho^i = 0$.

Next, we apply supersymmetry transformations to the double copy relations (2.20) and (2.21), to infer the double copy relations for the remaining supergravity fields. This will be discussed at length in Appendix D, to which we refer. For the combinations given in (2.15) we obtain the double copy relations

$$\begin{aligned} aT_{\mu\nu}^- &= -4\sigma \star \tilde{F}_{\mu\nu}^- , \\ b\mathcal{F}_{\mu\nu}^{0+} &= -\sigma \star \tilde{F}_{\mu\nu}^+ , \end{aligned} \quad (2.24)$$

and hence

$$\begin{aligned} F^{0+} &= -\frac{1}{b} \sigma \star \tilde{F}^+ - \frac{2\langle X^0 \rangle}{\bar{a}} \bar{\sigma} \star \tilde{F}^+ , \\ F^{1+} &= \frac{\langle z \rangle}{b} \sigma \star \tilde{F}^+ - \frac{2\langle X^1 \rangle}{\bar{a}} \bar{\sigma} \star \tilde{F}^+ . \end{aligned} \quad (2.25)$$

This yields the following relations for the supergravity field strengths $F_{\mu\nu}^I$,

$$\begin{aligned} F_{\mu\nu}^0 &= -\left(\frac{\langle \bar{X}^0 \rangle}{a} + \frac{1}{2b}\right) \sigma \star \tilde{F}_{\mu\nu} - \left(\frac{1}{2b} - \frac{\langle \bar{X}^0 \rangle}{a}\right) \sigma \star (*\tilde{F})_{\mu\nu} + h.c. , \\ F_{\mu\nu}^1 &= \left(-\frac{\langle \bar{X}^1 \rangle}{a} + \frac{\langle z \rangle}{2b}\right) \sigma \star \tilde{F}_{\mu\nu} + \left(\frac{\langle z \rangle}{2b} + \frac{\langle \bar{X}^1 \rangle}{a}\right) \sigma \star (*\tilde{F})_{\mu\nu} + h.c. . \end{aligned} \quad (2.26)$$

Note that the expressions for F^0 and F^1 get interchanged under

$$\langle X^0 \rangle \leftrightarrow \langle X^1 \rangle , \quad \frac{1}{b} \leftrightarrow -\frac{\langle z \rangle}{b} . \quad (2.27)$$

More generally, we note that the symplectic transformation $\langle X^0 \rangle \rightarrow \kappa \langle X^1 \rangle$, $\langle X^1 \rangle \rightarrow \langle X^0 \rangle / \kappa$, together with $1/b \leftrightarrow -\langle z \rangle \kappa / b$ (with $\kappa \in \mathbb{R}$), interchanges F^0 and F^1 and preserves the prepotential $F = -iX^0X^1$.

We now summarize the resulting on-shell double copy dictionary for all the supergravity fields. Reinstating the dependence on the spectator field $\phi_{a\bar{a}}$, it is given by

$$\boxed{\begin{aligned} a R_{\mu\nu\alpha\beta}^- &= -\frac{1}{2} \left[F_{\mu\nu}^a \star \phi_{a\bar{a}} \star \tilde{F}_{\alpha\beta}^{\bar{a}-} + F_{\alpha\beta}^{a-} \star \phi_{a\bar{a}} \star \tilde{F}_{\mu\nu}^{\bar{a}} - 4\eta_{[\alpha[\mu} \partial_{\nu]} \partial_{\beta]}^- A^{a\rho} \star \phi_{a\bar{a}} \star \tilde{A}_{\rho}^{\bar{a}} \right] \\ a \psi_{\mu\nu}^i &= \varepsilon^{ij} \gamma^\rho \gamma_{[\nu} \lambda_j^a \star \phi_{a\bar{a}} \star \partial_{\mu]} \tilde{A}_{\rho}^{\bar{a}} \\ a T_{\mu\nu}^- &= -4\sigma^a \star \phi_{a\bar{a}} \star \tilde{F}_{\mu\nu}^{\bar{a}-} \\ b \mathcal{F}_{\mu\nu}^{0+} &= -\sigma^a \star \phi_{a\bar{a}} \star \tilde{F}_{\mu\nu}^{\bar{a}+} \\ b \partial_\mu \Omega^{0i} &= \frac{1}{2} \varepsilon^{ij} \gamma^\rho \lambda_j^a \star \phi_{a\bar{a}} \star \partial_\mu \tilde{A}_{\rho}^{\bar{a}} \\ b \partial_\mu \bar{X}^0 &= \frac{1}{2} F_{\mu\rho}^{a-} \star \phi_{a\bar{a}} \star \tilde{A}^{\bar{a}\rho} \end{aligned}} \quad (2.28)$$

We note that in the expression for the Riemann tensor, the anti self-dual part is taken over the indices $\alpha\beta$. A completely equivalent expression is, of course, obtained if, instead, we take the anti self-dual part over $\mu\nu$. The double copy relation for Ω^{1i} follows from the one for Ω^{0i} by virtue of the relation (2.14), and the double copy relations for $F_{\mu\nu}^{\pm I}$ are as in (2.26), with the spectator field reinserted. Similarly, the dictionary for $\partial_\mu X^1$ follows immediately from that for $\partial_\mu X^0$, when we use (B.12). Observe that the on-shell dictionary (2.28) is invariant under local Abelian transformations $A \rightarrow A + d\alpha$, $\tilde{A} \rightarrow \tilde{A} + d\tilde{\alpha}$ by virtue of the equations of motion $\partial^\mu F_{\mu\nu} = 0$ and $\square \tilde{\alpha} = 0$, which follows from the Lorentz gauge condition.

We note that the expression for the Riemann tensor can also be written as

$$a R_{\mu\nu\alpha\beta}^- = 2 \left[F_{[\alpha[\mu}^a \star \phi_{a\bar{a}} \star \tilde{F}_{\nu]\beta]}^{\bar{a}} + \eta_{[\alpha[\mu} \partial_{\nu]} \partial_{\beta]} A^{a\rho} \star \phi_{a\bar{a}} \star \tilde{A}_{\rho}^{\bar{a}} \right]^- . \quad (2.29)$$

If we restrict the parameter a to $a = a_{\mathbb{R}} \in \mathbb{R}$, then we are left with a simpler expression for the Riemann tensor,

$$a_{\mathbb{R}} R_{\mu\nu\alpha\beta} = -\frac{1}{2} \left[F_{\mu\nu}^a \star \phi_{a\tilde{a}} \star \tilde{F}_{\alpha\beta}^{\tilde{a}} + F_{\alpha\beta}^a \star \phi_{a\tilde{a}} \star \tilde{F}_{\mu\nu}^{\tilde{a}} - 4\eta_{[\alpha[\mu} \partial_{\nu]} A^{a\rho} \star \phi_{a\tilde{a}} \star \tilde{A}_{\rho}^{\tilde{a}} \right] . \quad (2.30)$$

It can be checked that the supersymmetry variation of the right hand side of (2.28) correctly induces the supergravity transformation of the left hand side, and vice-versa, by means of the double copy dictionary. The double copy relations in (2.28) are also consistent with the equations of motion of all the fields involved. Hence, we have established a consistent on-shell double copy dictionary for this model.

The on-shell double copy dictionary is formulated in terms of field strengths. However, one could give a double copy prescription in terms of fields, but in doing so one must be very careful in making consistent gauge choices, in particular, when peeling off derivatives in (2.28).

3 Dyonic BPS black hole solutions

In the following we consider dyonic BPS black hole solutions in the model $F = -iX^0 X^1$, and we verify that they have a double copy description based on the double copy relations given in (2.28).

On the supergravity side, the BPS conditions are derived by imposing the restriction [35]

$$k \epsilon_i = \varepsilon_{ij} \gamma_0 \epsilon^j , \quad (3.1)$$

where k denotes a phase factor with an appropriate chiral weight, so that both sides have the same chiral weight. We impose the same condition on the field theory side.

To keep the expressions as simple as possible, we again omit non-Abelian indices (a, \tilde{a}) , and only reinstate their dependence at the end.

3.1 Field theory side

On the field theory side, we seek static BPS solutions to the rigid supersymmetry transformations displayed in (2.9),

$$\delta_Q \lambda_i = \gamma^\mu \partial_\mu \sigma \epsilon_i + \frac{1}{4} \gamma^{\mu\nu} F_{\mu\nu}^- \varepsilon_{ij} \epsilon^j = 0 , \quad (3.2)$$

where we have suppressed the non-Abelian index a , as mentioned above. We impose the BPS condition (3.1), which results in

$$\begin{aligned} \partial_0(\sigma \bar{k}) &= 0 , \\ F_{mn}^- + 4\partial_{[m}^-(\sigma \bar{k}) \eta_{n]0} &= 0 \quad , \quad m, n = 1, 2, 3 , \end{aligned} \quad (3.3)$$

where the superscript ‘-’ denotes the anti-selfdual part. The second equation results in

$$F_{mn} + 2 \left[2\partial_{[m} \text{Re}(\sigma \bar{k}) \eta_{n]0} - \varepsilon_{mnp0} \partial^p \text{Im}(\sigma \bar{k}) \right] = 0 . \quad (3.4)$$

We restrict to static BPS configurations supported by electric fields only. In Cartesian coordinates, the BPS configurations we consider are thus described by

$$\begin{aligned}\partial_t(\sigma\bar{k}) &= 0 , \\ \partial_i \text{Im}(\sigma\bar{k}) &= 0 , \\ F_{ti} &= -2\partial_i \text{Re}(\sigma\bar{k}) , \\ F_{ij} &= 0 \quad , \quad i, j = x, y, z .\end{aligned}\tag{3.5}$$

In a spherically symmetric context, the electric potential $\text{Re}(\sigma\bar{k})$ will only depend on the radial coordinate r . When comparing with the supergravity BPS solutions, we will find it convenient to work with coordinates (u, r, θ, ϕ) , with u given by $u = t + r$. The resulting metric $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$ are given in (E.15). The gauge potential A_u reads

$$A_u = 2\text{Re}(\sigma\bar{k}) .\tag{3.6}$$

Remarkably, this electric BPS configuration will be mapped to a dyonic BPS black hole configuration in the following.

3.2 Supergravity side

The model $F = -iX^0X^1$ admits dyonic BPS black hole solutions, as first pointed out in [37]. These solutions, briefly reviewed in Appendix E, are supported by two electric charges, q_0 and q_1 , and by two magnetic charges p^0 and p^1 . We find it convenient to work in Eddington-Finkelstein type coordinates (u, r, θ, ϕ) , defined in (E.10). We linearise the solution around a flat background of the form (2.11), where $\eta_{\mu\nu}$ is given by (E.15), and the background scalar fields are

$$\langle X^0 \bar{k} \rangle = -\frac{1}{2} (h_1 - ih^0) \quad , \quad \langle X^1 \bar{k} \rangle = -\frac{1}{2} (h_0 - ih^1) \quad , \quad \langle z \rangle = \frac{h_0 - ih^1}{h_1 - ih^0} ,\tag{3.7}$$

where h_0, h_1, h^0, h^1 are constants entering the definition of the harmonic functions appearing in the attractor equations (E.4). As shown in Appendix E, the fluctuating fields are then given by

$$\begin{aligned}h_{\mu\nu} &= \text{diag} \left(\frac{Q}{r}, 0, Qr, Qr \sin^2 \theta \right) , \\ F_{ur}^0 &= \frac{Qh_1 - q_1}{r^2} \quad , \quad F_{ur}^1 = \frac{Qh_0 - q_0}{r^2} \quad , \quad F_{\theta\phi}^I = p^I \sin \theta \quad , \quad I = 0, 1 , \\ X^0 \bar{k} &= -\frac{(q_1 - ip^0 - \frac{1}{2}Q(h_1 - ih^0))}{2r} ,\end{aligned}\tag{3.8}$$

where

$$Q = h_0q_1 + h_1q_0 + h^0p^1 + h^1p^0 .\tag{3.9}$$

The expression for the fluctuation $X^1 \bar{k}$ follows from the one for $X^0 \bar{k}$ by interchanging the indices 0 and 1, and is related to $X^0 \bar{k}$ by (B.12).

3.3 Double copy expressions

Now we show that the supergravity fluctuations (3.8) have a double copy description based on the dictionary (2.28) by determining the associated gauge field configuration satisfying (3.5). To this end, we will employ (2.30) (we thus take the parameter a to be real) to write down a gauge fixed double copy expression for the fluctuating metric $h_{\mu\nu}$ (suppressing non-Abelian indices)

$$a h_{\mu\nu} = A_\mu \star \tilde{A}_\nu + A_\nu \star \tilde{A}_\mu - \left(A_\alpha \star \tilde{A}^\alpha \right) \eta_{\mu\nu} . \quad (3.10)$$

Note that the double copy relations were derived in the Cartesian coordinate system. Here, we work with spherical coordinates (u, r, θ, ϕ) , for convenience. In this coordinate system, the convolution integral for fields reads

$$\left[\phi \star \tilde{\phi} \right] (x) = \int \phi(y) \tilde{\phi}(x-y) \sqrt{-\eta(y)} d^4 y , \quad (3.11)$$

while for field strengths we use

$$\sigma \star \tilde{F}_{MN} = 2 \partial_{[M} \left(\sigma \star \tilde{A}_{N]} \right) . \quad (3.12)$$

Further, note that the convolution of a function with the dual of a tensor, $\sigma \star (*\tilde{F})$ is implemented as the dual of the convolution of the function with the tensor, $*(\sigma \star \tilde{F})$, which is consistent with the Cartesian implementation.

We take the non-vanishing components of A_μ and \tilde{A}_μ to be

$$\begin{aligned} A_u &= \frac{d_1}{r} , \\ \tilde{A}_u &= \delta^{(4)}(u, r, \theta, \phi) \quad , \quad \tilde{A}_r = d_2 \delta^{(4)}(u, r, \theta, \phi) , \\ \delta^{(4)}(u - u_0, r - r_0, \theta - \theta_0, \phi - \phi_0) &= \delta(u - u_0) \delta(r - r_0) \delta(\theta - \theta_0) \delta(\phi - \phi_0) / (r^2 \sin \theta) , \end{aligned} \quad (3.13)$$

with constant coefficients d_1 and d_2 which we now determine. Inserting this into (3.10) we find the relations

$$\begin{aligned} a Q &= d_1 (2 + d_2) , \\ a Q &= -d_1 d_2 , \end{aligned} \quad (3.14)$$

from which we infer

$$\begin{aligned} d_1 &= a Q , \\ d_2 &= -1 . \end{aligned} \quad (3.15)$$

Then, using (3.5) and (3.6), we take

$$2 \text{Re}(\sigma \bar{k}) = \frac{d_1}{r} \quad , \quad \text{Im}(\sigma \bar{k}) = 0 . \quad (3.16)$$

This, together with (3.13), determines the field configuration on the gauge theory side. We now show that it correctly reproduces the fluctuating fields (3.8).

We first consider the double copy description of the fluctuating scalar field X^0 . Using (E.18) we obtain

$$\bar{k} \partial_\mu X^0 = -\frac{(q_1 - ip^0 - \frac{1}{2}Q(h_1 - ih^0))}{2} \partial_\mu \frac{1}{r}. \quad (3.17)$$

On the other hand, using the double copy relations (2.28) for the electric field configuration (3.13), we infer,

$$\bar{k} \partial_\nu X^0 = \frac{\bar{k}}{4b} \partial_\nu (A^\alpha \star \tilde{A}_\alpha) = \frac{\bar{k}}{4b} \partial_\nu \frac{d_1 d_2}{r} = -\frac{\bar{k} a Q}{4b} \partial_\nu \frac{1}{r}. \quad (3.18)$$

Comparing with (3.17) we obtain

$$\frac{\bar{k} a}{2b} = \frac{q_1 - ip^0 - \frac{1}{2}Q(h_1 - ih^0)}{Q}. \quad (3.19)$$

Note that under the interchange of the 0-sector and the 1-sector, this transforms as in (2.27) by virtue of the BPS relation $h^I q_I = h_I p^I$, as it should.

Next we employ (3.19) as well as (3.16) in the double copy relations for the field strengths $F_{\mu\nu}^I$ using (2.26). First we consider $F_{\mu\nu}^0$. Using the result that the only non-vanishing integral in (2.26) is $\sigma \star \tilde{F}_{ur}$, we obtain

$$\begin{aligned} F_{ur}^0 &= -\left(\frac{\langle \bar{X}^0 \rangle}{a} + \frac{1}{2b}\right) \sigma \star \tilde{F}_{ur} + h.c. \\ &= -\left(\frac{\langle \bar{X}^0 k \rangle}{a} + \frac{k}{2b}\right) \partial_r \left(\sigma \bar{k} \star (-\tilde{A}_u)\right) + h.c. \\ &= \frac{(Qh_1 - q_1)}{aQ} \frac{d_1}{r^2} = \frac{Qh_1 - q_1}{r^2}, \\ F_{\theta\phi}^0 &= i\varepsilon_{\theta\phi}{}^{ur} \left(-\frac{\langle \bar{X}^0 \rangle}{a} + \frac{1}{2b}\right) \sigma \star \tilde{F}_{ur} + h.c. \\ &= -2\varepsilon_{\theta\phi}{}^{ur} \text{Im} \left[\left(-\frac{\langle \bar{X}^0 k \rangle}{a} + \frac{k}{2b}\right) \right] \sigma \bar{k} \star \tilde{F}_{ur} \\ &= \frac{2p^0}{aQ} r^2 \sin \theta \partial_r \left(\sigma \bar{k} \star (-\tilde{A}_u)\right) \\ &= \frac{p^0}{aQ} r^2 \sin \theta \frac{d_1}{r^2} = p^0 \sin \theta, \end{aligned} \quad (3.20)$$

in agreement with (3.8).

Now we consider $F_{\mu\nu}^1$ and compute

$$\begin{aligned} \text{Re} \left(-\frac{\langle \bar{X}^1 k \rangle}{a} + \frac{\langle z k \rangle}{2b} \right) &= \frac{Q h_0 - q_0}{aQ}, \\ \text{Im} \left(\frac{\langle z k \rangle}{2b} + \frac{\langle \bar{X}^1 k \rangle}{a} \right) &= -\frac{p^1}{aQ}, \end{aligned} \quad (3.21)$$

where we made use of the normalization condition (E.7) as well as of the BPS constraint

$h^I q_I = h_I p^I$. We obtain

$$\begin{aligned}
F_{ur}^1 &= 2Re \left(-\frac{\langle \bar{X}^1 k \rangle}{a} + \frac{\langle z k \rangle}{2b} \right) \sigma \bar{k} \star \tilde{F}_{ur} \\
&= 2Re \left(-\frac{\langle \bar{X}^0 z k \rangle}{a} + \frac{\langle z \rangle k}{2b} \right) \partial_r \left(\sigma \bar{k} \star (-\tilde{A}_u) \right) \\
&= \frac{(Q h_0 - q_0) d_1}{a Q r^2} = \frac{Q h_0 - q_0}{r^2} , \\
F_{\theta\phi}^1 &= -i\varepsilon_{\theta\phi}{}^{ur} \left(\frac{\langle \bar{X}^1 \rangle}{a} + \frac{\langle z \rangle}{2b} \right) \sigma \star \tilde{F}_{ur} + h.c. \\
&= 2\varepsilon_{\theta\phi}{}^{ur} Im \left[\left(\frac{\langle \bar{X}^1 k \rangle}{a} + \frac{\langle z \rangle k}{2b} \right) \right] \sigma \bar{k} \star \tilde{F}_{ur} \\
&= \frac{2p^1}{a Q} r^2 \sin \theta \partial_r \left(\sigma \bar{k} \star (-\tilde{A}_u) \right) \\
&= \frac{p^1}{a Q} r^2 \sin \theta \frac{d_1}{r^2} = p^1 \sin \theta , \tag{3.22}
\end{aligned}$$

in agreement with (3.8). Thus, we have verified that the double copy gauge field configuration (3.13) and (3.16) correctly reproduces the gravitational configuration (3.8).

We may easily reinstate the dependence on the non-Abelian indices, by taking $A_\mu^a = A_\mu c^a$, $\tilde{A}_\mu^a = \tilde{A}_\mu \tilde{c}^a$, $\phi_{a\tilde{a}} = V_{a\tilde{a}} \delta^{(4)}(u, r, \theta, \phi)$, with constant $c^a, \tilde{c}^a, V_{a\tilde{a}}$ normalised to $c^a V_{a\tilde{a}} \tilde{c}^{\tilde{a}} = 1$.

Finally, we observe that the configuration (3.13) satisfies the constraint (2.5). Hence, we conclude that in the weak field approximation, the dyonic BPS black hole solution has a double copy description, based on (2.28), in terms of an electrically charged BPS solution in gauge theory.

4 Conclusions

We have constructed the on-shell double copy dictionary for a $D = 4, \mathcal{N} = 2$ supergravity theory with one vector multiplet based on the prepotential $F = -iX^0 X^1$. In doing so, we made use of the explicit S-supersymmetry gauge fixing condition (2.14). For a different one vector multiplet prepotential, such as $F = -(X^1)^3/X^0$, this condition will look different. Thus, the double copy dictionary obtained here only applies to this particular prepotential.

Note that the double copy dictionary (2.28) was derived for source free theories. However, we showed that it also holds for a class of gravitational BPS configurations that have sources.

An important feature of the double copy construction is that the field configurations in the $\mathcal{N} = 2$ and $\mathcal{N} = 0$ sectors are constrained by

$$\partial^\mu \left(\varphi_{SYM}^a \star \phi_{a\tilde{a}} \star \tilde{A}_\mu^{\tilde{a}} \right) = 0 . \tag{4.1}$$

This relation also imposes a constraint on the gauge transformations of the field \tilde{A}_μ^a in the $\mathcal{N} = 0$ sector. Although this relation appears to couple fields in the two sectors, field

configurations where \tilde{A}_μ^a obeys a Lorentz like gauge condition (2.7) trivially satisfy this constraint, keeping the fields in the two sectors independent.

In this note we have also given a double copy description of all BPS single center black holes in the model based on the prepotential $F = -iX^0X^1$. Most interestingly, this description maps dyonic black holes to purely electric configurations on the field theory side. An obvious generalization consists in introducing a dyonic configuration in the field theory side and examining the corresponding configuration in gravity.

The on-shell double copy construction for $\mathcal{N} = 2$ supergravity theories with more than one vector multiplet will bring in new ingredients. Namely, adding vector multiplets on the supergravity side will require adding scalar fields in the $\mathcal{N} = 0$ sector of the double copy [19]. We plan to address this in the near future.

Acknowledgements

We would like to thank Sergei Alexandrov, Alexandros Anastasiou, Paolo Benincasa, Marco Chiodaroli, Michael Duff, Roberto Emparan, Ricardo Monteiro, Michele Zoccali for helpful discussions. This work was supported by FCT/Portugal through grant EXCL/MAT-GEO/0222/2012 (G.L.Cardoso), through a CAMGSD post-doc fellowship (S. Nagy) and through FCT fellowship SFRH/BPD/101955/2014 (S. Nampuri). This work was also supported by the COST action MP1210 "The String Theory Universe". G.L.C. would like to thank the Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institute) and the University of the Witwatersrand for kind hospitality during various stages of this work.

A Conventions

We follow the conventions of [36]. We usually denote spacetime indices by μ, ν, \dots , frame indices by $a, b, \dots = 0, 1, 2, 3$ and $SU(2)$ R-symmetry indices by $i, j, \dots = 1, 2$. We use the following (anti-)symmetrization conventions,

$$[a, b] = \frac{1}{2}(ab - ba) \quad , \quad (ab) = \frac{1}{2}(ab + ba) . \quad (\text{A.1})$$

We take

$$\gamma_a \gamma_b = \eta_{ab} + \gamma_{ab} \quad , \quad \gamma_{ab} = \frac{1}{2}[\gamma_a, \gamma_b] , \quad (\text{A.2})$$

where $\eta_{ab} = \text{diag}(-, +, +, +)$. Introducing $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$, we define projection operators in the usual way,

$$P_L = \frac{1}{2}(\mathbb{I} + \gamma_5) , \quad P_R = \frac{1}{2}(\mathbb{I} - \gamma_5) . \quad (\text{A.3})$$

The chirality assignment of a chiral fermion is specified by the position of the $SU(2)$ R-symmetry index, for instance

$$\begin{aligned} \psi_{\mu i} &= P_R \psi_{\mu i} \quad , \quad \psi_\mu^i = P_L \psi_\mu^i , \\ \Omega^{Ii} &= P_R \Omega^{Ii} \quad , \quad \Omega_i^I = P_L \Omega_i^I , \\ \epsilon_i &= P_R \epsilon_i \quad , \quad \epsilon^i = P_L \epsilon^i . \end{aligned} \quad (\text{A.4})$$

Under h.c., an $SU(2)$ R-symmetry index changes position.

The complete antisymmetric tensor ε_{abcd} satisfies $\varepsilon_{0123} = 1$. The dual of an antisymmetric tensor field F_{ab} is given by

$$(*F)_{ab} = -\frac{i}{2} \varepsilon_{abcd} F^{cd} . \quad (\text{A.5})$$

The (anti-)selfdual part of F_{ab} is determined by

$$F_{ab}^{\pm} = \frac{1}{2} (F_{ab} \pm (*F)_{ab}) . \quad (\text{A.6})$$

We note the relations

$$\begin{aligned} \gamma_{ab} &= \frac{i}{2} \varepsilon_{abcd} \gamma^{cd} \gamma_5 , \\ \gamma^{ab} F_{ab} \epsilon^i &= \gamma^{ab} F_{ab}^- \epsilon^i , \end{aligned} \quad (\text{A.7})$$

as well as

$$\gamma_\rho \gamma^{\alpha\beta} F_{\alpha\beta}^- (P_L \chi) = -4 \gamma^\alpha F_{\alpha\rho}^- (P_L \chi) . \quad (\text{A.8})$$

B Supergravity transformation rules

We work within the superconformal approach for $\mathcal{N} = 2$ supergravity coupled to $\mathcal{N} = 2$ vector multiplets [25–29]. We summarise some of its features that are relevant for this paper. In the Poincaré frame, after eliminating the auxiliary fields, the fields are the spacetime metric $g_{\mu\nu}$, the gravitini ψ_μ^i , the gauge fields W_μ^I , the gaugini Ω_i^I and the scalar fields X^I . They transform as follows under Q -supersymmetry (dropping higher-order fermionic terms),

$$\begin{aligned} \delta_Q g_{\mu\nu} &= \bar{\epsilon}^i \gamma_{(\mu} \psi_{\nu)i} + h.c. , \\ \delta_Q \psi_\mu^i &= \mathcal{D}_\mu \epsilon^i - \frac{1}{16} T_{ab}^- \gamma^{ab} \gamma_\mu \varepsilon^{ij} \epsilon_j , \\ \delta_Q W_\mu^I &= \frac{1}{2} \varepsilon^{ij} \bar{\epsilon}_i (\gamma_\mu \Omega_j^I + 2 \psi_{\mu j} X^I) + h.c. , \\ \delta_Q \Omega_i^I &= \not{D} \bar{X}^I \epsilon^i + \frac{1}{4} \gamma^{ab} \mathcal{F}_{ab}^{I+} \varepsilon^{ij} \epsilon_j , \\ \delta_Q X^I &= \frac{1}{2} \bar{\epsilon}^i \Omega_i^I , \end{aligned} \quad (\text{B.1})$$

where the covariant derivatives are given by

$$\begin{aligned} \mathcal{D}_\mu \bar{X}^I &= (\partial_\mu + i a_\mu) \bar{X}^I , \\ \mathcal{D}_\mu \epsilon^i &= \left(\partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} - \frac{i}{2} a_\mu \right) \epsilon^i , \end{aligned} \quad (\text{B.2})$$

and we have the composite quantities

$$\begin{aligned} T_{ab}^- &= 2 \frac{N_{IJ} \bar{X}^J}{N_{KL} \bar{X}^K \bar{X}^L} F_{ab}^{I-} , \\ \mathcal{F}_{ab}^{I+} &= F_{ab}^{I+} - \frac{1}{2} X^I T_{ab}^+ , \\ a_\mu &= -\frac{1}{2} (F_I \partial_\mu \bar{X}^I - \bar{X}^I \partial_\mu F_I + \text{c.c.}) . \end{aligned} \quad (\text{B.3})$$

Here $F_I = \partial F(X)/\partial X^I$, where $F(X)$ denotes the prepotential of the model. The scalar fields X^I satisfy the Einstein frame constraint

$$N_{IJ} X^I \bar{X}^J = -1 , \quad (\text{B.4})$$

where

$$N_{IJ} = -i (F_{IJ} - \bar{F}_{IJ}) \quad , \quad F_{IJ} = \frac{\partial^2 F(X)}{\partial X^I \partial X^J} . \quad (\text{B.5})$$

The gaugini Ω_i^I are constrained by the S-supersymmetry gauge fixing condition

$$\bar{X}^I N_{IJ} \Omega_i^J = 0 . \quad (\text{B.6})$$

Now we focus on the model $F(X) = -iX^0 X^1$, for which $N_{00} = N_{11} = 0$, $N_{01} = N_{10} = -2$, as well as

$$\Omega_i^1 = -\bar{z} \Omega_i^0 , \quad (\text{B.7})$$

where $z = X^1/X^0$. The Einstein frame constraint (B.4) becomes

$$2 |X^0|^2 (z + \bar{z}) = 1 . \quad (\text{B.8})$$

Next, we linearise this theory around a flat spacetime background with metric $\eta_{\mu\nu}$ and constant scalar fields $\langle X^I \rangle$,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} , \quad X^I = \langle X^I \rangle + \delta X^I . \quad (\text{B.9})$$

In order to avoid cluttering of notation, we will denote the fluctuations δX^I simply by X^I . The linearised spin connection $\omega_{\mu ab}$ and the linearised Riemann tensor $R_{\mu\nu ab} = \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab}$ are given by (dropping pure gauge terms in $\omega_{\mu ab}$)

$$\begin{aligned} \omega_{\mu ab} &= -\partial_{[a} h_{b]\mu} , \\ R_{\mu\nu\alpha\beta} &= -2\partial_{[\alpha} \partial_{[\mu} h_{\nu]\beta]} . \end{aligned} \quad (\text{B.10})$$

The Q-supersymmetry parameter ϵ^i splits into a local part and a global (rigid) part. The S-supersymmetry constraint (B.7) results in

$$\partial_\mu (\Omega_i^1 + \langle \bar{z} \rangle \Omega_i^0) = 0 , \quad (\text{B.11})$$

which, upon contraction with $\bar{\epsilon}^i$, equals the supersymmetry variation of

$$\partial_\mu (X^1 + \langle \bar{z} \rangle X^0) = 0 . \quad (\text{B.12})$$

The Einstein frame constraint (B.4) reduces to

$$X^0 \langle \bar{X}^1 \rangle + X^1 \langle \bar{X}^0 \rangle + \langle X^0 \rangle \bar{X}^1 + \langle X^1 \rangle \bar{X}^0 = \frac{1}{2} . \quad (\text{B.13})$$

Using this, the connection a_μ in (B.3) becomes

$$a_\mu = -2i \langle \bar{X}^0 \rangle (\partial_\mu X^1 + \langle \bar{z} \rangle \partial_\mu X^0) , \quad (\text{B.14})$$

which vanishes by virtue of (B.12). Hence, at the linearised level, we have

$$a_\mu = 0 . \quad (\text{B.15})$$

For the other composite fields in (B.3) we obtain, at the linearised level,

$$\begin{aligned} T_{\mu\nu}^- &= \frac{1}{\langle \bar{X}^0 \rangle} \left[F_{\mu\nu}^{-0} + \frac{F_{\mu\nu}^{-1}}{\langle \bar{z} \rangle} \right] , \\ \mathcal{F}_{\mu\nu}^{0+} &= \frac{1}{2} \left[F_{\mu\nu}^{0+} - \frac{F_{\mu\nu}^{+1}}{\langle z \rangle} \right] , \\ \mathcal{F}_{\mu\nu}^{1+} &= \frac{1}{2} [F_{\mu\nu}^{1+} - \langle z \rangle F_{\mu\nu}^{+0}] . \end{aligned} \quad (\text{B.16})$$

C Equations of motion

In this appendix we summarise all the equations of motion of the fields involved, both on the gauge side and on the gravity side. Since we work in the linearised approximation, we note that the equations for the fluctuations of different fields decouple. We also describe the gauge fixing of the non-supersymmetryc field \tilde{A}_μ .

C.1 (Super) Yang-Mills

The equations of motion and Bianchi identities for the fields of the $\mathcal{N} = 2$ super Yang-Mills multiplet, in the absence of sources, are

$$\begin{aligned} \partial_\mu F^{\mu\nu a} &= \partial_\mu (*F)^{\mu\nu a} = 0 , \\ \not{\partial} \lambda_i^a &= 0 , \\ \square \sigma^a &= 0 , \end{aligned} \quad (\text{C.1})$$

and similarly for the source-free $\mathcal{N} = 0$ Yang-Mills gauge field,

$$\partial_\mu \tilde{F}^{\mu\nu \tilde{a}} = \partial_\mu (*\tilde{F})^{\mu\nu \tilde{a}} = 0 . \quad (\text{C.2})$$

We will find that it is useful to work in a Lorenz like gauge

$$\partial_\mu \tilde{A}^{\mu \tilde{a}} = 0 . \quad (\text{C.3})$$

In this gauge the associated equation of motion reduces to

$$\square \tilde{A}_\mu^{\tilde{a}} = 0 . \quad (\text{C.4})$$

C.2 Supergravity

C.2.1 Fermionic fields

In the linearised theory, the gaugini satisfy the Dirac equation

$$\not{\partial} \Omega^{Ii} = 0 , \quad (\text{C.5})$$

and the gravitini satisfy

$$\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho^i = 0, \quad (\text{C.6})$$

which can be brought into the equivalent form

$$\gamma^\mu[\partial_\mu\psi_\nu^i - \partial_\nu\psi_\mu^i] = \gamma^\mu\psi_{\mu\nu}^i = 0. \quad (\text{C.7})$$

Applying ∂_ρ to this and antisymmetrizing in $\rho\nu$ results in

$$\not\partial\psi_{\rho\nu}^i = 0. \quad (\text{C.8})$$

Finally, by contracting (C.7) with $\gamma^\rho\partial_\rho$, we obtain yet another form,

$$\partial^\mu\psi_{\mu\nu}^i = 0. \quad (\text{C.9})$$

C.2.2 Bosonic fields

The scalar fluctuations will satisfy the wave equation

$$\square X^I = 0, \quad (\text{C.10})$$

and the gauge fields will decouple to individually satisfy the Maxwell equation and Bianchi identity

$$\partial^\mu F_{\mu\nu}^I = \partial^\mu(*F)_{\mu\nu}^I = 0. \quad (\text{C.11})$$

Finally, in the weak field limit, Einstein's equations reduce to

$$R_{\mu\nu} = 0. \quad (\text{C.12})$$

D Dictionary derivation

In this section we describe how the double copy dictionary in (2.28) is obtained. We will use the following simplified notation throughout this appendix,

$$\varphi^a \star \phi_{a\bar{a}} \star \tilde{\varphi}^{\bar{a}} \equiv \varphi \star \tilde{\varphi}. \quad (\text{D.1})$$

This is motivated by the fact that the transformation properties of the spectator scalar do not contribute to the derivation of the dictionary. We work in the Lorentz like gauge $\partial_\mu \tilde{A}^\mu = 0$. Additionally, we will make extensive use of the following property of the convolution

$$\partial_\mu(f \star g) = (\partial_\mu f) \star g = f \star (\partial_\mu g). \quad (\text{D.2})$$

In Table 1, we present the on-shell tensoring of helicity states. Motivated by this, we begin with the ansatz

$$a\psi_{\mu\nu}^i + 2b\gamma_{[\nu}\partial_{\mu]}\Omega^{0i} \equiv \varepsilon^{ij}\lambda_j \star \tilde{F}_{\mu\nu}, \quad (\text{D.3})$$

with $a, b \in \mathbb{C}$ complex constants, carrying the appropriate chiral weights under $U(1)$ (we note that the chiral weights of the gravitini and the gaugini differ by ± 1 , depending on

convention). We now contract the above with γ^μ and, making use of the gravitini equation of motion (C.7), we get

$$2b\gamma^\mu\gamma_{[\nu}\partial_{\mu]}\Omega^{0i} = \varepsilon^{ij}\gamma^\mu\lambda_j \star (\partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu) . \quad (\text{D.4})$$

Now we use the Clifford algebra relation (A.2) together with the equation of motion for the gaugini (C.5) on the left hand side (LHS); on the right hand side (RHS) we employ the property (D.2), together with the equation of motion for λ_j from (C.1), to simplify the expression to

$$2b\partial_\mu\Omega^{0i} = \varepsilon^{ij}\gamma^\rho\lambda_j \star \partial_\mu\tilde{A}_\rho . \quad (\text{D.5})$$

Thus, we have obtained the dictionary entry for the gaugini. The next step is to check whether this double copy expression satisfies the equation of motion for the gaugini (C.5). To verify this, we contract (D.5) with γ^μ to get

$$\begin{aligned} 0 &= 2b\gamma^\mu\partial_\mu\Omega^{0i} = \varepsilon^{ij}\gamma^\mu\gamma^\rho\lambda_j \star \partial_\mu\tilde{A}_\rho \\ &= \varepsilon^{ij}\gamma^\rho\cancel{\partial}\lambda_j \star \tilde{A}_\rho + 2\varepsilon^{ij}\lambda_j \star \partial^\mu\tilde{A}_\mu , \end{aligned} \quad (\text{D.6})$$

where to get to the second line we made use of the Clifford algebra relation (A.2) and the convolution property (D.2). The first term now vanishes by (C.1), while the second vanishes because \tilde{A}_μ is taken to satisfy the Lorenz like gauge (C.3). Finally, the LHS of (D.5) must vanish when we impose the equation of motion for \tilde{A}_μ . Working with the form $\square\tilde{A}_\mu = 0$, we see that this is indeed the case, since

$$\square\Omega_i^I = 0 \quad (\text{D.7})$$

in the linearised theory. Next, we plug in the dictionary for the gaugini into (D.3) to read off the dictionary for the gravitini,

$$a\psi_{\mu\nu}^i = \varepsilon^{ij}[2\partial_{[\mu}\lambda_j \star \tilde{A}_{\nu]} - \gamma_{[\nu}\gamma^\rho\lambda_j \star \partial_{\mu]}\tilde{A}_\rho] . \quad (\text{D.8})$$

We can make use of the Clifford algebra relation to rewrite this in the simpler form

$$a\psi_{\mu\nu}^i = \varepsilon^{ij}\gamma^\rho\gamma_{[\nu}\lambda_j \star \partial_{\mu]}\tilde{A}_\rho . \quad (\text{D.9})$$

As before, we now proceed to checking whether this dictionary is compatible with the equation of motion (C.7). Contracting the expression above with γ^μ and using $\gamma^b\gamma_a\gamma_b = -2\gamma_a$ we obtain

$$a\gamma^\mu\psi_{\mu\nu}^i = \frac{1}{2}\varepsilon^{ij}\left(2\gamma_\nu\lambda_j \star \partial^\mu\tilde{A}_\mu + \gamma^\rho\gamma_\nu\cancel{\partial}\lambda_j \star \tilde{A}_\rho\right) , \quad (\text{D.10})$$

where we made use of the convolution property (D.2). This vanishes by virtue of (C.1) and because \tilde{A}_μ is taken to satisfy the Lorentz like gauge (C.3). Moreover, it is easy to see from the above that putting the Yang-Mills fields λ_i and \tilde{A}_μ on-shell exactly corresponds to putting the gravitini on-shell.

We now proceed to derive the dictionary for the field strengths of the supergravity gauge fields W_μ^I . It is convenient to first work out double copy expressions for the combinations $T_{\mu\nu}^-$ and $\mathcal{F}_{\mu\nu}^{0+}$, given in (2.15), that appear in the linearized supergravity transformation rules

(2.13). To derive $T_{\mu\nu}^-$, we will make use of the supersymmetry variation of the gravitini field strength, by comparing terms of like chirality or, equivalently, of the same helicity. Using (2.13), we obtain

$$a\delta_Q\psi_{\mu\nu}^i(T) = -\frac{1}{8}\partial_{[\mu}(aT_{\alpha\beta}^-)\gamma^{\alpha\beta}\gamma_{\nu]}\varepsilon^{ij}\epsilon_j. \quad (\text{D.11})$$

Here, the notation $\delta_Q\psi_{\mu\nu}^i(T)$ means that we are only considering terms in the variation that are proportional to T . Similarly, when writing $\delta_Q\lambda_j(\sigma)$ below, we are only retaining terms proportional to σ . This has to match with the supersymmetry variation on the super Yang-Mills side,

$$\begin{aligned} a\delta_Q\psi_{\mu\nu}^i(T) &= \varepsilon^{ij}\gamma^\rho\gamma_{[\nu}(\delta_Q\lambda_j(\sigma))\star\partial_{\mu]}\tilde{A}_\rho \\ &= \gamma^\rho\gamma_{[\nu}\gamma^\alpha\partial_\alpha\sigma\star\partial_{\mu]}\tilde{A}_\rho\varepsilon^{ij}\epsilon_j \\ &= -\gamma^\rho\gamma^\alpha\partial_\alpha\sigma\star\partial_{[\mu}\tilde{A}_\rho\gamma_{\nu]}\varepsilon^{ij}\epsilon_j \\ &= -\gamma^\rho\gamma^\alpha\partial_{[\mu}(\sigma\star\partial_\alpha\tilde{A}_\rho)\gamma_{\nu]}\varepsilon^{ij}\epsilon_j. \end{aligned} \quad (\text{D.12})$$

Here we got to the second line via (2.9), to the third line via the Clifford algebra relations, and the final expression is obtained through the convolution property (D.2). Using the relation (A.2) once more as well as the Lorenz gauge condition $\partial^\rho\tilde{A}_\rho = 0$, we are left with

$$\begin{aligned} a\delta_Q\psi_{\mu\nu}^i(T) &= \frac{1}{2}\gamma^{\rho\alpha}\partial_{[\mu}(\sigma\star\tilde{F}_{\rho\alpha})\gamma_{\nu]}\varepsilon^{ij}\epsilon_j \\ &= \frac{1}{2}\gamma^{\alpha\beta}\partial_{[\mu}(\sigma\star\tilde{F}_{\alpha\beta}^-)\gamma_{\nu]}\varepsilon^{ij}\epsilon_j, \end{aligned} \quad (\text{D.13})$$

where in the last line we have projected out the self-dual part, in light of (A.7). Finally, comparing (D.11) and (D.13), we read off

$$aT_{\mu\nu}^- = -4\sigma\star\tilde{F}_{\mu\nu}^-. \quad (\text{D.14})$$

The other composite quantity, $\mathcal{F}_{\mu\nu}^{0+}$, is derived analogously from the supersymmetry variation of the gaugini. Using (2.13), we have

$$2b\delta_Q\partial_\mu\Omega^{0i}(\mathcal{F}) = \frac{1}{2}\gamma^{\alpha\beta}\partial_\mu(b\mathcal{F}_{\alpha\beta}^{0+})\varepsilon^{ij}\epsilon_j. \quad (\text{D.15})$$

We match this with the supersymmetry variation on the super Yang-Mills side,

$$\begin{aligned} 2b\delta_Q\partial_\mu\Omega^{0i}(\mathcal{F}) &= \varepsilon^{ij}\gamma^\rho\delta_Q\lambda_j(\sigma)\star\partial_\mu\tilde{A}_\rho \\ &= \gamma^\rho\gamma^\alpha\partial_\alpha\sigma\star\partial_\mu\tilde{A}_\rho\varepsilon^{ij}\epsilon_j \\ &= \gamma^\rho\gamma^\alpha\partial_\mu(\sigma\star\partial_\alpha\tilde{A}_\rho)\varepsilon^{ij}\epsilon_j, \end{aligned} \quad (\text{D.16})$$

where we used (2.9) to get to the second line and the non-Leibniz behaviour of the convolution (D.2) to get to the third line. We again use the relation (A.2) and the Lorenz gauge for \tilde{A}_ρ to write

$$\begin{aligned} 2b\delta_Q\partial_\mu\Omega^{0i}(\mathcal{F}) &= -\frac{1}{2}\gamma^{\rho\alpha}\partial_\mu(\sigma\star\tilde{F}_{\rho\alpha})\varepsilon^{ij}\epsilon_j \\ &= -\frac{1}{2}\gamma^{\alpha\beta}\partial_\mu(\sigma\star\tilde{F}_{\alpha\beta}^+)\varepsilon^{ij}\epsilon_j, \end{aligned} \quad (\text{D.17})$$

where we made use of the right-handed version of (A.7). Then, comparing (D.15) and (D.17), we get

$$b\mathcal{F}_{\mu\nu}^{0+} = -\sigma \star \tilde{F}_{\mu\nu}^+ . \quad (\text{D.18})$$

We now recall, from (2.15), that

$$\begin{aligned} T_{\mu\nu}^- &= \frac{1}{\langle \bar{X}^0 \rangle} \left[F_{\mu\nu}^{0-} + \frac{F_{\mu\nu}^{1-}}{\langle \bar{z} \rangle} \right] , \\ \mathcal{F}_{\mu\nu}^{0+} &= \frac{1}{2} \left[F_{\mu\nu}^{0+} - \frac{F_{\mu\nu}^{1+}}{\langle z \rangle} \right] , \end{aligned} \quad (\text{D.19})$$

from which we extract

$$\begin{aligned} F_{\mu\nu}^0 &= -\sigma \star \left[\frac{2\langle \bar{X}^0 \rangle}{a} \tilde{F}_{\mu\nu}^- + \frac{1}{b} \tilde{F}_{\mu\nu}^+ \right] + h.c. , \\ F_{\mu\nu}^1 &= -\sigma \star \left[\frac{2\langle \bar{X}^1 \rangle}{a} \tilde{F}_{\mu\nu}^- - \frac{\langle z \rangle}{b} \tilde{F}_{\mu\nu}^+ \right] + h.c. . \end{aligned} \quad (\text{D.20})$$

Given that $\tilde{F}_{\mu\nu}^\pm = \frac{1}{2}(\tilde{F}_{\mu\nu} \pm (*\tilde{F})_{\mu\nu})$, it is now easy to see that the equations of motion and Bianchi identities for the field strengths $F_{\mu\nu}^I$ are in direct correspondence with those of the Yang-Mills side. One can also check that the LHS and RHS of the above expressions transform identically under supersymmetry.

We continue with the derivation of the dictionary for the supergravity scalar X^0 . We make use of the supersymmetry transformation of the gaugini (2.13) to write

$$2b\delta_Q \partial_\mu \Omega^{0i}(X) = 2b\gamma^\rho \partial_\mu (\partial_\rho \bar{X}^0) \epsilon^i . \quad (\text{D.21})$$

We will compare this with the supersymmetry variation on the super Yang-Mills side,

$$\begin{aligned} 2b\delta_Q \partial_\mu \Omega^{0i}(X) &= \varepsilon^{ij} \gamma^\rho \delta_Q \lambda_j(F) \star \partial_\mu \tilde{A}_\rho \\ &= -\frac{1}{4} \gamma_\rho \gamma^{\alpha\beta} F_{\alpha\beta}^- \star \partial_\mu \tilde{A}^\rho \epsilon^i , \end{aligned} \quad (\text{D.22})$$

where we plugged in (2.9) and used $\varepsilon^{ij} \varepsilon_{jk} = -\delta_k^i$. We now make use of the relation (A.8) to rewrite

$$\begin{aligned} 2b\delta_Q \partial_\mu \Omega^{0i}(X) &= \gamma^\alpha F_{\alpha\rho}^- \star \partial_\mu \tilde{A}^\rho \epsilon^i \\ &= \gamma^\rho \partial_\mu (F_{\rho\nu}^- \star \tilde{A}^\nu) \epsilon^i , \end{aligned} \quad (\text{D.23})$$

where we renamed dummy variables and made use of the convolution property (D.2). Then, comparing (D.21) and (D.23), we read off the dictionary for the scalar

$$b\partial_\mu \bar{X}^0 = \frac{1}{2} F_{\mu\rho}^- \star \tilde{A}^\rho . \quad (\text{D.24})$$

As before, we proceed by checking the equations of motion. This is done most easily by contracting the expression above with ∂^μ ,

$$b\Box \bar{X}^0 = \frac{1}{2} \partial^\mu F_{\mu\rho}^- \star \tilde{A}^\rho . \quad (\text{D.25})$$

Thus we see that the equation of motion for the scalar field follows from the equation of motion and Bianchi identity for the super Yang-Mills gauge field, and vice-versa. Similarly, imposing the equations of motion for \tilde{A}_μ implies the equation of motion for X^0 . Additionally, one can show that both sides of the above equation transform identically under supersymmetry. Also, acting with ∂_ν on (D.24) and anti-symmetrising in $\mu\nu$, one obtains the relation

$$0 = F_{\rho[\mu}^- \star \partial_{\nu]} \tilde{A}^\rho . \quad (\text{D.26})$$

This relation is satisfied by virtue of the equations of motion and the Lorentz like gauge for \tilde{A}_μ .

The final step is to derive the dictionary for the Riemann tensor. We recall that

$$R_{\mu\nu\alpha\beta} = -2\partial_{[\alpha}\partial_{[\mu}h_{\nu]\beta]} , \quad (\text{D.27})$$

and we make use of the supersymmetry transformation of the gravitino (2.13),

$$a\delta_Q\psi_{\mu\nu}^i(R) = \frac{a}{4}\gamma^{\alpha\beta}R_{\mu\nu\alpha\beta}^-\epsilon^i , \quad (\text{D.28})$$

where the anti self-dual part is taken over $\alpha\beta$. This is compared with the supersymmetry variation on the super Yang-Mills side (2.9),

$$\begin{aligned} a\delta_Q\psi_{\mu\nu}^i(R) &= \varepsilon^{ij}\gamma^\rho\gamma_{[\nu}(\delta_Q\lambda_j(F))\star\partial_{\mu]}\tilde{A}_\rho \\ &= -\frac{1}{4}\gamma^\rho\gamma_{[\nu}\gamma^{\alpha\beta}F_{\alpha\beta}^-\star\partial_{\mu]}\tilde{A}_\rho\epsilon^i . \end{aligned} \quad (\text{D.29})$$

We now make use of (A.8) to simplify the above to

$$a\delta_Q\psi_{\mu\nu}^i(R) = \gamma^\rho\gamma^\alpha F_{\alpha[\nu}^-\star\partial_{\mu]}\tilde{A}_\rho\epsilon^i . \quad (\text{D.30})$$

At this stage we again use $\gamma^\alpha\gamma^\beta = \eta^{\alpha\beta} + \gamma^{\alpha\beta}$, and, in light of (D.26), we are left with

$$a\delta_Q\psi_{\mu\nu}^i(R) = \gamma^{\rho\alpha}F_{\alpha[\nu}^-\star\partial_{\mu]}\tilde{A}_\rho\epsilon^i . \quad (\text{D.31})$$

We now recall the definition of the anti self-dual tensor from (A.6), and making judicious use of the gamma matrix identities in (A.7), together with the equations of motion for the super Yang-Mills gauge field, we derive

$$a\delta_Q\psi_{\mu\nu}^i(R) = -\frac{1}{8}\gamma^{\alpha\beta}\left[F_{\mu\nu}\star\tilde{F}_{\alpha\beta}^- + F_{\alpha\beta}^-\star\tilde{F}_{\mu\nu} - 4\eta_{[\alpha[\mu}\partial_{\nu]}\partial_{\beta]}^-A^\rho\star\tilde{A}_\rho\right] . \quad (\text{D.32})$$

Finally, we compare (D.28) and (D.32), and read off

$$aR_{\mu\nu\alpha\beta}^- = -\frac{1}{2}\left[F_{\mu\nu}\star\tilde{F}_{\alpha\beta}^- + F_{\alpha\beta}^-\star\tilde{F}_{\mu\nu} - 4\eta_{[\alpha[\mu}\partial_{\nu]}\partial_{\beta]}^-A^\rho\star\tilde{A}_\rho\right] . \quad (\text{D.33})$$

This can also be written as

$$aR_{\mu\nu\alpha\beta}^- = 2\left[F_{[\alpha[\mu}\star\tilde{F}_{\nu]\beta]} + \eta_{[\alpha[\mu}\partial_{\nu]}\partial_{\beta]}^-A^\rho\star\tilde{A}_\rho\right]^- , \quad (\text{D.34})$$

where the anti self-dual part is taken over the $\alpha\beta$ indices. We can then extract the double copy relation for the Riemann tensor,

$$R_{\mu\nu\alpha\beta} = -\frac{1}{2a} \left[F_{\mu\nu} \star \tilde{F}_{\alpha\beta}^- + F_{\alpha\beta}^- \star \tilde{F}_{\mu\nu} - 4\eta_{[\alpha[\mu} \partial_{\nu]} \partial_{\beta]}^- A^\rho \star \tilde{A}_\rho \right] + h.c. . \quad (\text{D.35})$$

Now we check that the Ricci tensor constructed from $R_{\mu\nu\alpha\beta}$ vanishes, hence that Einstein's equations (C.12) are satisfied. One can show that this holds as a consequence of equations of motion and Bianchi identities on the field theory side. Moreover, we have checked that the LHS and RHS of the above equation transform identically under supersymmetry. Finally, it is easy to show that the Riemann tensor satisfies the Bianchi identity $R_{\mu[\nu\alpha\beta]} = 0$, as required.

E Dyonic BPS black hole solutions

We consider dyonic BPS black hole solutions in the model based on $F(X) = -iX^0X^1$. These solutions are supported by electric charges (q_0, q_1) and by magnetic charges (p^0, p^1) [37].

The associated line element is of the form

$$ds^2 = -e^{2g} dt^2 + e^{-2g} (dr^2 + r^2 d\Omega_2^2) , \quad (\text{E.1})$$

with $g = g(r)$ determined by

$$e^{-2g} = i(\bar{Y}^I F_I(Y) - Y^I \bar{F}_I(\bar{Y})) . \quad (\text{E.2})$$

Here $F_I(Y) = \partial F(Y)/\partial Y^I$, with the Y^I defined by [38]

$$Y^I = e^{-g} X^I \bar{k} , \quad (\text{E.3})$$

where k denotes the compensating phase introduced in (3.1). The Y^I are determined by the attractor equations

$$\begin{aligned} Y^I - \bar{Y}^I &= iH^I , \\ F_I(Y) - \bar{F}_I(\bar{Y}) &= iH_I , \end{aligned} \quad (\text{E.4})$$

where the (H_I, H^I) denote harmonic functions

$$H_I = h_I + \frac{q_I}{r}, \quad H^I = h^I + \frac{p^I}{r} , \quad I = 0, 1 , \quad (\text{E.5})$$

with integration constants $h_I \in \mathbb{R}, h^I \in \mathbb{R}$ that satisfy the BPS constraint $h^I q_I = h_I p^I$ [35]. We obtain

$$Y^0 = -\frac{1}{2} (H_1 - iH^0) , \quad Y^1 = -\frac{1}{2} (H_0 - iH^1) , \quad e^{-2g} = H_0 H_1 + H^0 H^1 . \quad (\text{E.6})$$

We impose the asymptotic normalization condition $e^{-2g}|_{r=\infty} = 1$, which results in

$$h_0 h_1 + h^0 h^1 = 1 . \quad (\text{E.7})$$

The black hole horizon is at $r = 0$. These black holes are supported by a complex scalar field,

$$z = \frac{Y^1}{Y^0} = \frac{H_0 - iH^1}{H_1 - iH^0} , \quad (\text{E.8})$$

and by electric-magnetic fields [38]

$$F_{tr}^I = -\partial_r (e^{2g} (Y^I + \bar{Y}^I)) \quad , \quad F_{\theta\phi}^I = p^I \sin \theta . \quad (\text{E.9})$$

We find it convenient to work with an Eddington-Finkelstein type coordinate u defined by

$$du = dt + e^{-2g} dr . \quad (\text{E.10})$$

In coordinates (u, r, θ, ϕ) , the line element (E.1) becomes

$$ds^2 = -e^{2g} du^2 + 2dudr + e^{-2g} r^2 d\Omega_2^2 . \quad (\text{E.11})$$

Next, we perform a weak field approximation of the solution. This will be used in the main text to obtain the double copy structure of this solution. At large r , e^{-2g} is approximated by

$$e^{-2g} = 1 + \frac{Q}{r} + \mathcal{O}(r^{-2}) , \quad (\text{E.12})$$

where we have introduced the notation

$$Q = h_0 q_1 + h_1 q_0 + h^0 p^1 + h^1 p^0 . \quad (\text{E.13})$$

The line element (E.11) becomes

$$ds^2 = -du^2 + 2dudr + r^2 d\Omega_2^2 + \frac{Q}{r} du^2 + Q r d\Omega_2^2 . \quad (\text{E.14})$$

The resulting metric is of form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with the background metric $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$ given by

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 1 & & \\ 1 & 0 & & \\ & r^2 & 0 & \\ & 0 & r^2 \sin^2 \theta & \end{pmatrix} , \quad \eta^{\mu\nu} = \begin{pmatrix} 0 & 1 & & \\ 1 & 1 & & \\ & \frac{1}{r^2} & 0 & \\ & 0 & \frac{1}{r^2 \sin^2 \theta} & \end{pmatrix} , \quad (\text{E.15})$$

and with the fluctuation metric $h_{\mu\nu}$ given by

$$h_{\mu\nu} = \text{diag} \left(\frac{Q}{r}, 0, Q r, Q r \sin^2 \theta \right) . \quad (\text{E.16})$$

The scalar fields X^0 and z get approximated by

$$X^0 \rightarrow \langle X^0 \rangle + X^0 \quad , \quad z \rightarrow \langle z \rangle + z , \quad (\text{E.17})$$

where on the right hand side X^0 and z denote fluctuating fields. Using (E.7), we obtain

$$\begin{aligned} \langle X^0 \bar{k} \rangle &= -\frac{1}{2} (h_1 - i h^0) \quad , \quad \langle z \rangle = \alpha , \\ X^0 \bar{k} &= -\frac{(q_1 - i p^0 - \frac{1}{2} Q (h_1 - i h^0))}{2r} \quad , \quad z = \frac{\Sigma}{r} , \end{aligned} \quad (\text{E.18})$$

where

$$\alpha = \frac{h_0 - ih^1}{h_1 - ih^0} \quad , \quad \Sigma = \frac{q_0 - ip^1 - \alpha(q_1 - ip^0)}{h_1 - ih^0} . \quad (\text{E.19})$$

Finally, the electric field strengths are approximated by

$$F_{ur}^0 = \frac{Qh_1 - q_1}{r^2} \quad , \quad F_{ur}^1 = \frac{Qh_0 - q_0}{r^2} , \quad (\text{E.20})$$

while the magnetic field strengths are exact and given in (E.9).

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